APPENDIX IIIB-A

DIFFERENTIATION OF S^{MB}

The entropy of a macrostate is defined by eqn. IIIB-1. It is specialized for Boltzons by substitution of the expression for the number of microstates in a Boltzon macrostate, eqn. IIIA-7a.

$$S_i^{MR} = k \ln W_i^{MB}; \quad \text{where} \quad W_i^{MB} = N! \prod_j \frac{g_j^{n_{j,i}}}{n_{j,i}!} \qquad \text{(IIIB-1,IIIA-7a)}$$

To derive the equilibrium distribution function, $n_{j,mp}^{MB}$, it is necessary to differentiate S_i^{MB} with respect to $n_{j,i}$ and simplify.

$$d\left(S_i^{MB}\right) = kd\left(\ln W_i^{MB}\right) = kd\left[\ln \left(N! \prod_j \frac{g_j^{n_{j,i}}}{n_{j,i}!}\right)\right]$$

Differentiation is simplified by expressing the logarithm of a product as the sum of the logarithms, viz.

$$\ln\left(\prod_{j=1} x_j\right) = \ln(x_1 \cdot x_2 \cdot x_3 \ldots) = \ln x_1 + \ln x_2 + \ln x_3 + \cdot + \cdot = \sum_j \ln x_j$$

Applied to $\ln W_i^{MB}$, this yields

$$\ln \left(N! \prod_j \frac{g_j^{n_{j,i}}}{n_{j,i}!}\right) = \ln (N!) + \sum_j [n_{j,i} \ln g_j - \ln (n_{j,i}!)]$$

This result can be further simplified by representing the factorials using *Stirling's approximation*

$$x! \approx (2\pi x)^{\frac{1}{2}} x^x e^{-x} \qquad \text{(IIIB-A1)}$$

Stirling's formula is accurate for arguments $x \geq 10$ (see Table IIIB-A1). But we need the logarithm of the factorial. The logarithm of eqn. IIIB-A1, viz., $\ln (x!) \approx [\ln (2\pi x]/2 + x \ln$ or $x - x$ provides the required result. But the first of these terms is small in comparison to the last two and can be omitted when $x \geq 10^2$.

$$\ln (x!) \approx x \ln x - x \qquad \text{(IIIB-A2)}$$

The accuracy of this approximation is illustrated in Table IIIB-A1.

TABLE IIIB-A1: THE ACCURACY OF STIRLING'S APPROXIMATION, eqns. IIIB-A1 and A2

x	x!	$\left(\frac{x! - SA}{x!}\right)\%$	$\ln(x!)$	$\frac{\ln(x!) - \ln SA}{\ln(x!)}$ %
10^1	$3.628 \cdot 10^9$	0.76	$1.510 \cdot 10^4$	13.8
10^2	$9.333 \cdot 10^{160}$	0.082	$3.637 \cdot 10^5$	0.887
10^3	$4.024 \cdot 10^{2570}$	0.0083	$5.914 \cdot 10^{16}$	0.105