$$\left(\frac{\partial S^{MB}_{eq}}{\partial U}\right)_{N,V} = k\beta + kU\left(\frac{\partial \beta}{\partial U}\right)_{V,N} + Nk\frac{\partial}{\partial \beta}\left[\ell n\ Z(\beta,V)\right]_V\left(\frac{\partial \beta}{\partial U}\right)_{V,N}$$

The last term is simplified as

$$\frac{\partial}{\partial \beta}\left[\ell n\ Z(\beta,V)\right]_V = \frac{1}{Z}\left(\frac{\partial Z}{\partial \beta}\right)_V = \frac{1}{Z}\sum_j g_j \frac{\partial}{\partial \beta}\left(e^{-\beta\varepsilon_j(V)}\right)_V = -\frac{1}{Z}\sum_j \varepsilon_j g_j e^{-\beta\varepsilon_j(V)}$$

Multiplying by N/N and introducing $n^{MB}_{j,mp} = \frac{N}{Z} g_j e^{-\beta\varepsilon_j(V)}$, gives

$$\frac{\partial}{\partial \beta}\left[\ell n\ Z(\beta,V)\right]_V = -\frac{1}{N}\sum_j n^{MB}_{j,mp}\ \varepsilon_j(V) = -\frac{U}{N}$$

So we have

$$\left(\frac{\partial S^{MB}}{\partial U}\right)_{N,V} = k\beta + kU\left(\frac{\partial \beta}{\partial U}\right)_{V,N} - Nk\frac{U}{N}\left(\frac{\partial \beta}{\partial U}\right)_{V,N} = k\beta, \qquad \text{QED.}$$

The preceding example established that the entropy changes of regions A and B are given as, $dS_A = k\beta_A dU_A$ and $dS_B = k\beta_B dU_B$. A first law analysis of these regions shows $\left(dU_A = đQ_A + \cancel{đW_A}\right) = -\left(dU_B = đQ_B + \cancel{đW_B}\right)$. Thus the entropy change of the system can be written

$$dS|_{isol} = dS_A + dS_B = kdQ_A(\beta_A - \beta_B) \geq 0$$

Examine the dimensions of this relationship. Both entropy and heat are extensive. Therefore, β must be a fundamentally intensive property. As equilibrium is achieved, the entropy reaches a maximum ($dS_{isol} = 0$) and $\beta_A \rightarrow \beta_B = \beta_{eq}$. β is a fundamentally intensive property that is uniform when a system is in thermal equilibrium. This is the same characteristic that is displayed by temperature. We conclude that $\beta = \beta(T)$. To determine the form of the relationship consider the expression developed above for the isolated system, dS_{isol}. If heat enters system A, $đQ_A > 0$, then the second law, $dS|_{isol} > 0$, requires that $\beta_A > \beta_B$. So heat flows from regions of low β toward regions of high β. This is opposite to the relationship of heat flow and temperature. The relationship between β and T is reciprocal, $\beta \propto 1/T$. We choose as our explicit relationship the simplest form that satisfies our objectives

$$\beta \equiv \frac{1}{kT} \qquad \text{(IIIB-10)}$$

Boltzmann's constant, k, is required to ensure the same numerical values for temperature as our previous scales (Unit IA).

The equilibrium distribution for Boltzons can now be written in terms of measurable parameters.