Then substituting the dimensionless function

$$x = \frac{hc}{\lambda kT} \qquad dx = \frac{-hc}{\lambda^2 kT}\,d\lambda \qquad \lambda = 0,\ x \to \infty \text{ and } \lambda \to \infty,\ x = 0$$

we obtain

$$s_{v,pht} = \left[\frac{8\pi^5 k^4}{15(hc)^3}\right]T^3 - 8\pi k\left(\frac{kT}{hc}\right)^3 \int_\infty^0 \left(\frac{hc}{kT}\right)^3 \left(\frac{-hc}{\lambda^2 kT}\,d\lambda\right) \ln\left[1 - e^{-(hc/\lambda kT)}\right]$$

$$= \frac{8\pi^5}{15}\,k\left(\frac{kT}{hc}\right)^3 - 8\pi k\left(\frac{kT}{hc}\right)^3 \int_0^\infty x^2 \ln(1 - e^{-x})\,dx$$

This integral can be transformed to the Debye integral form using "by parts" integration. To that purpose, define

$$W = \ln(1 - e^{-x}) \qquad dW = \frac{e^{-x}}{1 - e^{-x}}\,dx = \frac{dx}{e^x - 1} \qquad dV = x^2 dx \qquad V = \frac{x^3}{3}$$

Thus

$$\int_0^\infty x^2 \ln(1 - e^{-x})\,dx = \frac{x^3}{3}\ln(1 - e^{-x})\Big|_0^\infty - \frac{1}{3}\int_0^\infty \frac{x^3}{e^x - 1}\,dx = -\frac{1}{3}\int_0^\infty \frac{x^3 dx}{e^x - 1} = -\frac{1}{3}\left(\frac{\pi^4}{15}\right)$$

Substituting, we obtain

$$s_{v,pht} = \frac{8\pi^5 k^4}{15(hc)^3}\,T^3\left(1 + \frac{1}{3}\right) = \frac{32\pi^5 k^4}{45(hc)^3}\,T^3$$

Comparing this result to eqn. IIID-7b, we see $s_{v,pht} = 4u_{v,pht}/3T$.

Example 4. An expression for the entropy of a photon gas was derived in the preceding example. Use this result in conjunction with the Gibbs equation, $dS = dU/T + (p/T)dV$, to derive an expression for radiation pressure.

The Gibbs equation is of the form $S = S(U,V)$. The definition of the derivative of a function of this form allows us to express pressure in terms of a partial derivative of the entropy

$$dS = \left(\frac{\partial S}{\partial U}\right)_V dU + \left(\frac{\partial S}{\partial V}\right)_U dV = \frac{dU}{T} + \frac{p}{T}\,dV \qquad \text{Hence} \quad p = T\left(\frac{\partial S}{\partial V}\right)_U$$

The total entropy is $S_{pht} = Vs_{v,pht} = V \cdot 4u_v/(3T)$. To facilitate the differentiation, we abbreviate

$$u_v = \left[\frac{8}{15}\,\frac{\pi^5 k^4}{(hc)^3}\right]T^4 = aT^4$$

Ans. 4. Wein's Law gives λ_{max}.

$$T = 300\text{ K}; \quad \lambda_{max} = \frac{2898\ \mu\text{m K}}{300\text{ K}} = 9.663\ \mu\text{m}$$

Similarly at: $T = 3000$ K, $\lambda_{max} = 0.9663\ \mu$m, $T = 30{,}000$ K, $\lambda_{max} = 0.09663\ \mu$m

Ans. 5. Presumably, the alien creatures' eyes have adapted to the maximum emission of their "sun." Thus,

$$T_{star} = \frac{2898\ \mu\text{m K}}{0.25\ \mu\text{m}} = 11{,}600\text{ K}$$