APPENDIX IIIC-A

FURTHER CONSIDERATION OF THE TRANSLATION MODE

Exmp. 4 develops Z_{tr} using the states formulation. This is the easiest and therefore the preferred method. But developments to be completed in Units IIID and IIIE require formulation of particle distribution over translational levels. Specifically, an analytic expression for the degeneracy, g_{tr}, is required. The following example uses the result of Exmp. 3 to deduce this function.

Example A1. Exmp. 4 formulated Z_{tr} in the limit $T >> \theta_{tr}$ as a triple integral over the quantum numbers. In a geometric sense the quantum numbers were the coordinates of a cartesian spatial description of the states and the integration was over the positive quadrant. The magnitude of an energy state, $\varepsilon(\ell,m,n)$, is proportional to the square of its position vector $(\ell^2 + m^2 + n^2)$. Thus states of the same energy are connected by spherical surfaces within the positive quadrant $(\ell,m,n > 0)$. The level index, q, is proportional to the magnitude of the energy. And the number of states per level is proportional to the area of the spherical segment. Thus $q^2 = \ell^2 + m^2 + n^2$ provides a direct analogy between the two representations. Derive an analytic expression for the degeneracy, $g_{tr}(q)$, using this analogy by transforming the cartesian states integral to a spherical level integral.

The triple integral over the states developed in Exmp. 4 is to be transformed under the definition $\ell^2 + m^2 + n^2 = q^2$. Thus the element of volume in the cartesion space transforms to $d\ell\, dm\, dn = q^2\cos\theta\, d\theta\, d\phi\, dq$. And while $1 \le n,m,\ell < \infty$ the new coordinates satisfy the limits: $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$, $\sqrt{3} \le q < \infty$. Thus

$$Z_{tr} = \left(\int_1^\infty e^{-\ell^2\theta_{tr}/T}\, d\ell\right)\left(\int_1^\infty e^{-m^2\theta_{tr}/T}\, dm\right)\left(\int_1^\infty e^{-n^2\theta_{tr}/T}\, dn\right)$$

$$= \int_0^{\pi/2} \cos\theta\, d\theta \int_0^{\pi/2} d\phi \int_{\sqrt{3}}^\infty e^{-q^2\theta_{tr}/T}\, q^2 dq = \frac{\pi}{2}\int_{\sqrt{3}}^\infty q^2\, e^{-q^2\theta_{tr}/T}\, dq$$

Comparing this result to eqn. IIIC-14c, we conclude, $g_{tr}(q) = \pi q^2/2$.

Exercise A1. Exmp. A1 formulated Z_{tr} for the limit $T >> \theta_{tr}$ in terms of an integral over the levels. Verify that this result is correct by completing the required integral (use Table IIIE-1) and comparing the result to that of Exmp. 4.

Exmp. 4 and Exer. A1 formulated the high-temperature limit translational partition function in two of its three possible integral forms, viz., as an integral over the states and levels. The third formulation is as an integral over the energy magnitudes themselves.

Exercise A2. Transform the level integral formulation of Exmp. 1 to an integral over the energy. Identify the degeneracy, $g_{tr}(\varepsilon)$.

Exer. A2 expressed $g_{tr}(\varepsilon)$ in terms of θ_{tr}. In Unit IIID, we will employ this function to determine the equilibrium properties of electromagnetic and acoustical radiation, and