Example B1. Substitute eqn. IIIA-7c into eqn. IIIB-1 and simplify using Stirling's approximation for the logarithm of a factorial (eqn. IIIB-A2) to develop S_i^{FD} as a sum of two terms in $n_{j,i}$ and g_j.

The logarithm of eqn. IIIA-7c can be expressed as a sum by using the rule for the logarithm of a product (see AppendixIIIB-A).

$$\ln W_i^{FD} = \sum_j [\ln (g_j!) - \ln [(g_j - n_{j,i})!] - \ln (n_{j,i}!)]$$

Then when $\ln x! = x \ln x - x$ (eqn. IIIB-A2) is used to approximate the logarithms, we obtain

$$\ln W_i^{FD} = \sum_j [g_j \ln g_j - g_j - (g_j - n_{j,i}) \ln (g_j - n_{j,i}) + (g_j - n_{j,i}) - n_{j,i} \ln n_{j,i} + n_{j,i}]$$

Cancelling and collecting terms we obtain the entropy, eqn. IIIB-B1 shown below.

The expression for the entropy of a Fermion macrostate was simplified in the preceding example. It is

$$S_i^{FD} = k \ln\left(W_i^{FD}\right) = k \sum_j \left[g_j \ln \left(\frac{g_j}{g_j - n_{j,i}}\right) + n_{j,i} \ln \left(\frac{g_j - n_{j,i}}{n_{j,i}}\right)\right] \qquad \text{(IIIB-B1)}$$

In the following example this expression is differentiated to derive the distribution function for a Fermi-Dirac system in equilibrium.

Example B2. Differentiate eqn. IIIB-B1 (at fixed N and V). Then use eqn. IIIB-7 to obtain the most probable distribution, $n_{j,mp}^{FD}$, in terms of α and β.

It is somewhat easier to differentiate the logarithms when they are written separately, see the last equation in Exmp. B1. Its differential is[20]

$$d\left(\frac{S_i^{FD}}{k}\right) = d\left(\ln W_i^{FD}\right) = \sum_j \left[+\ln (g_j - n_{j,i}) - \ln \left(n_{j,i}\right) + \frac{(g_j - n_{j,i})}{(g_j - n_{j,i})} - \frac{n_{j,i}}{n_{j,i}}\right] \delta n_{j,i} = \sum_j \left[\ln \left(\frac{g_j - n_{j,i}}{n_{j,i}}\right)\right] \delta n_{j,i}$$

Then substituting into eqn. IIIB-7, we obtain

$$\sum_j \left[\ln \left(\frac{g_j - n_{j,i}}{n_{j,i}}\right) - \alpha - \beta\epsilon_j(V)\right] \delta n_{j,i} = 0$$

But since $\delta n_{j,i}$ are linearly independent, each coefficient separately satisfies

$$\ln \left(\frac{g_j - n_{j,mp}}{n_{j,mp}}\right) = \alpha + \beta\epsilon_j(V)$$

And taking the antilogarithm we obtain eqn. IIIB-B2, see below.

The equilibrium distribution for Fermions is

$$n_{j,mp}^{FD} = \frac{g_j}{e^{\alpha + \beta\epsilon_j(V)} + 1} \qquad \text{(IIIB-B2)}$$

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[20] g_j is a constant.