Application to the second term gives

$$\bar{c}_{v,phn} = 9\bar{R}\left[+\frac{T}{x_D^3}\frac{x_D^3}{(e^{x_D}-1)}\frac{d\left(\frac{\theta_D}{T}\right)}{dT}\right] = 3\bar{R}\left[4D(x_D) - \frac{3x_D}{(e^{x_D}-1)}\right]$$

Exercise 12. Eqn. IIIB-17 was adapted to determine the photon entropy density in Exmp. 3. A similar procedure is needed to derive eqn. IIID-16 for the phonon entropy. Adapt the procedure of Exmp. 3 to the specific case of phonons. Show that the result is eqn. IIID-16.

Exercise 13. Determine the lattice contribution to the internal energy, specific heat, and entropy of aluminum at 100, 300, and 1000 K.

Fig. IIID-3 shows the Debye model for the specific heat of phonons. Notice the similarity of this function to the modes of a molecular gas (see the low-and high-temperature limit summaries, Unit IIIC, Sec. IIIC). In the high-temperature, classical limit, $T \gg \theta_D$, the phonon gas is fully excited and $\bar{c}_{v,phn} = 6 \cdot (\bar{R}/2)$. In this low-temperature, quantum limit, $T \ll \theta_D$, the phonon gas is unexcited and the specific heat approaches zero. The form of this low temperature asymptote is one of the improvements that the Debye model makes over the Einstein model. Specifically (see Exmp. 7)

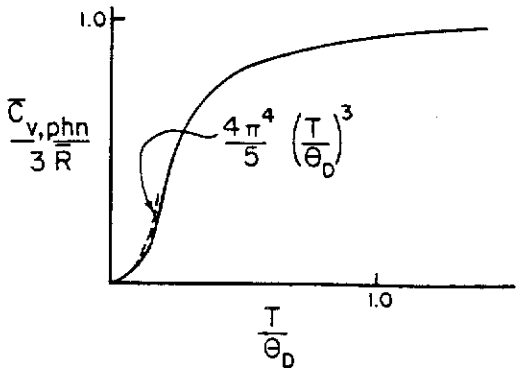

Fig. IIID-3 The Debye Specific Heat

$$\bar{c}_{v,phn} = \frac{12}{5}\pi^4\,\bar{R}\left(\frac{T}{\theta_D}\right)^3 \quad \text{for} \quad \frac{T}{\theta_D} \lesssim \frac{1}{12} \qquad \text{(IIID-17)}$$

Example 7. Determine the low-and high-temperature asymptotes of the Debye integral, eqn IIID-14. Use these limits to formulate the low-and high-temperature asymptotes of the phonon energy, specific heat, and entropy.

Ans. 11. The necessary densities and molecular weights are found in Tables ID-2 and IA-2, respectively. Thus

$$c_{s,Fe} = \frac{k\theta_D}{h}\left(\frac{4\pi M}{3\rho N_A}\right)^{1/3} = \frac{1.380 \cdot 10^{-23}\,\text{J/(part K)} \cdot 455\,\text{K}}{6.625 \cdot 10^{-34}\,\text{Js/part}}$$

$$\cdot \left[\frac{4\pi \cdot 55.85\,\text{g/(g mol)} \cdot 10^{-6}\,\text{m}^3/\text{cm}^3}{3 \cdot 7.88\,\text{g/cm}^3 \cdot 6.023 \cdot 10^{23}\,\text{part/(g mol)}}\right]^{1/3} = 3.48 \cdot 10^3\,\text{m/s}$$

Similarly for lead, using $\theta_D = 86$ K, $\rho = 11.34$ g/cm^3, $M = 207.2$ g/(g mol), one obtains, $c_{s,Pb} = 901$ m/s.