container. Notice that small values of particle mass and container volume tend to increase θ_{tr}. The following exercise investigates the limit.

Exercise 2. Hydrogen is the lightest pure substance. Determine the volume of a hydrogen container which would be required to achieve a value of θ_{tr} as high as the lowest temperatures that have been achieved in the laboratory, $\sim 10^{-6}$K.

The preceding problems demonstrate that gases composed of *atomic size particles are always in the high-temperature translational limit,* $T \gg \theta_{tr}$.[7] Thus, it is only necessary to evaluate Z_{tr} for this limit. Application of eqn. IIIC-14c requires that the degeneracy be expressed as an analytic function of the level index, $g_{tr}(\ell)$. But the translational energy magnitudes have so far been written in terms of state designation, i.e., the quantum numbers. Thus, we must either transform eqn. IIIC-14c to an integral over the states or derive an expression for the degeneracy in terms of the level index, $g_{tr}(\ell)$, or energy, $g_{tr}(\varepsilon)$. In Exmp. 4 we use the first, and simplest, method to derive Z_{tr}. Appendex IIIC-A demonstrates that the other two methods yield the same result and obtains expressions for g_{tr} for later reference.

Example 4. Formulate the alternative, state representation of the partition function for the high temperature limit (i.e., in analogy to eqn. IIIC-14c). Evaluate the required integrals (using Table IIIE-1) to derive an analytic expression for $Z_{tr}(T,\theta_{tr})$ valid in the limit, $T \gg \theta_{tr}$.

The series representation is a sum over the three quantum number indices. In the high-temperature limit, $T \gg \theta_{tr}$, these sums transform to a triple integral.

$$Z_{tr} = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-(\ell^2+m^2+n^2)\theta_{tr}/T} = \int_{\ell=1}^{\infty}\int_{m=1}^{\infty}\int_{n=1}^{\infty} e^{-(\ell^2+m^2+n^2)\theta_{tr}/T}\, d\ell\, dm\, dn$$

Using the property of the exponential ($e^{a+b} = e^a e^b$), each of these integrals is separable

$$Z_{tr} = \left[\int_1^{\infty} e^{-\ell^2(\theta_{tr}/T)} d\ell\right]\cdot\left[\int_1^{\infty} e^{m^2(-\theta_{tr}/T)} dm\right]\cdot\left[\int_1^{\infty} e^{n^2(-\theta_{tr}/T)} dn\right] = \left[\int_1^{\infty} \zeta^2(-\theta_{tr}/T) d\zeta\right]^3$$

$$= \left[\int_0^{\infty} e^{\zeta^2(-\theta_{tr}/T)} d\zeta - \int_0^{1} e^{\zeta^2(-\theta_{tr}/T)} d\zeta\right]^3$$

The second of these two integrals can be expressed as an error function

$$\int_0^1 e^{\zeta^2(-\theta_{tr}/T)} d\zeta = 2\left(\frac{T}{\pi\theta_{tr}}\right)^{\frac{1}{2}} \operatorname{erf}\left[\left(\frac{\theta_{tr}}{T}\right)^{\frac{1}{2}}\right]$$

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[7]In Unit IIID we consider translation of subatomic particles— electrons, photons, and phonons. We will find that even the mass of an electron is sufficient that free electrons are usually within the high temperature limit for translation, $T \gg \theta_{tr,el}$.

Ans. 1. $f_{tr}(\ell,m,n) = (\ell^2+m^2+n^2)$, $B_{tr} = h^2/(8m^*V^{2/3})$, and $\theta_{tr} = h^2/(8m^*V^{2/3}k)$.