We have

$$u_v(T) = 8\pi hc\left(\frac{kT}{hc}\right)^4 \int_0^\infty \frac{\left(\frac{hc}{\lambda^2 kT}d\lambda\right)\left(\frac{hc}{\lambda kT}\right)^3}{[e^{(hc/\lambda kT)}-1]} = \frac{8\pi(kT)^4}{(hc)^3}\int_\infty^0 \frac{-x^3dx}{(e^x-1)} = \frac{8\pi k^4}{(hc)^3}\left[\int_0^\infty \frac{x^3dx}{(e^x-1)}\right]T^4$$

Exmp. 2 evaluated eqn. IIID-7a in terms of a definite integral. The general (indefinite limit) form of this function is called the Debye integral; its values are listed in Table IIID-2 at the end of the Unit. Its value for the infinite limit is $\pi^4/15$ (see Ref. 5, Appendix E). This gives the total energy density as

$$u_{v,pht}(T) = \frac{8\pi^5 k^4}{15(hc)^3}T^4 \qquad \text{(IIID-7b)}$$

The number of photons per unit volume, or the *photon density* is developed in a similar manner, see Exer. 6. The result is

$$n_{v,pht}(T) = \frac{N_{pht}}{V} = \frac{1}{V}\int_0^\infty n_{pht,\lambda}(T)d\lambda = 8\pi \cdot 2.404\left(\frac{k}{hc}\right)^3 T^3 \qquad \text{(IIID-8)}$$

Exercise 6. Substitute the photon distribution into eqn. IIID-4b and express the required integration in terms of the dimensionless variable, x ($\equiv hc/\lambda kT$).

Exercise 7. Expressions for photon properties are normalized with volume rather than mass or mole since photons are massless. For purposes of comparison, one can normalize photon energy with Avogadro's number of photons and obtain photon energy in molar units. One should not, however, compare such "molar" photon properties with the molar properties of the matter that occupies the same space, see Exer. 9. The physical bases of these two molar quantities are not the same. Formulate an expression for the internal energy "per mole of photons," $\underline{u}_{pht}$. Differentiate this expression to derive an expression for $\underline{c}_{v,pht}$.

The energy and particle density of a photon gas were determined by direct application of their defining integrals. The following problems use the same procedures to determine the entropy density and pressure of a photon gas.

Note to the Student. The following examples develop expressions for the entropy and pressure of the photon gas. The results are reported as eqns. IIID-9 and 10. The techniques to develop these results are not representative of objectives of the unit.

Example 3. The entropy of a system of independent Bosons is described by eqn. IIIB-17.

$$S^{BE} = \frac{U}{T} + Nk\alpha - k\sum_{j=1}^{J} g_j \ln\left(1 - e^{-\alpha-\varepsilon_j/kT}\right)$$

Adapt this relationship to the specific case of a photon gas. Incorporate the continuum approximation by replacing the sum by an integral. Express the result as an integral over the dimensionless variable, x ($\equiv hc/\lambda kT$).

For the photon gas we can substitute: $\alpha = 0$, $\varepsilon_j = h\nu_j$, and $U = [8\pi^5k^4/15(hc)^3]VT^4$. The continuum approximation allows us to replace the sum over the energy levels by an integral over wavelength. Substituting into eqn. IIIB-17, we obtain

$$S^{BE} = \left[\frac{8\pi^5k^4}{15(hc)^3}\right]VT^4 - k\,8\pi V\int_0^\infty \frac{d\lambda}{\lambda^4}\ln\left[1 - e^{-(hc/\lambda kT)}\right]$$